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Abstract

In this paper, a free rigid body of dimension three is extended and analysed both in classical and quantum mechanics. The extension is performed by bringing the inverse inertia tensor, which is a positive-definite symmetric matrix for the ordinary rigid body, into an arbitrary real symmetric one. With an arbitrary real symmetric matrix chosen, associated is a Lie-Poisson structure on the Euclidean space of dimension three, through which the classical dynamics for an extended free rigid body is defined, and characterized by two first integrals. In parallel to this, the quantum dynamics is formulated as the problem of simultaneous spectral resolution of the two operators which are viewed as the quantization of the two classical first integrals. Intensive use is made of the unitary representation theory for Lie groups concerned. The explicit spectral resolution is obtained, in particular, when the extended free rigid body is an extended free symmetric top.

Key words: Extended rigid body, Lie-Poisson structure, Representation of three-dimensional Lie groups
2000 MSC: 17B63, 43A65, 70E15, 70E40, 70H05

1. Introduction

The present paper deals with an extension of the three-dimensional free rigid body dynamics both in classical and quantum mechanics.

In [4], D. D. Holm and J. E. Marsden show that ordinary free rigid body mechanics can be extended to Lie-Poisson mechanics on the three-dimensional Euclidean space $\mathbb{R}^3$ which is endowed with a family of compatible Lie-Poisson structures other than the standard one. These Lie-Poisson structures make the equations of motion

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for free rigid bodies into bi-Hamiltonian systems. The bi-Hamiltonian structures given to the Euler equations for the (higher-dimensional) free rigid bodies are introduced in [2, 3] independently of [4] and are intensively used for the proof of the involution of the integrals and the complete integrability of the systems. See also [8]. In the present paper, the extended free rigid body dynamics is introduced after [4], and its quantization is studied. Although the classical mechanics for an extended free rigid body seems to be understandable immediately through the bi-Hamiltonian structures, the quantum mechanics of it requires another deep study. The classical dynamics considered in this paper is also an example of the generalized Hamiltonian dynamics defined by Y. Nambu [9]. In his paper, Nambu discussed the quantization of the generalized Hamiltonian dynamics in the Heisenberg picture, but no solutions were found. In the present paper, the extended free rigid body is quantized in the Schrödinger picture, and dealt with from the representation-theoretic point of view. It then turns out that for an extended free symmetric top which admits an additional $SO(2)$-symmetry, solutions are found both in classical and quantum dynamics. To be precise, the eigenvalues are given explicitly for the extended free symmetric top in quantum mechanics, but no wave function is given.

The present article consists of two main sections, Sec.2 and Sec.3. In Sec.2, the classical dynamics for an extended free rigid body is dealt with. With an arbitrarily chosen real symmetric 3×3 matrix H, the Lie algebra $\mathbb{R}^3_{\mathsf{H}}$ is associated as in [4]. The classical extended free rigid body is defined to be a Lie-Poisson system on the dual space ${\mathbb{R}^3_{\mathsf{H}}}^*$ endowed with the canonical Lie-Poisson structure. The equation of motion takes the form

$$\frac{dp}{dt} = \nabla H \times \nabla L,$$

where $p \in {\mathbb{R}^3_{\mathsf{H}}}^*$, $H(p) = \dfrac{1}{2} p \cdot \mathsf{H} p$, and $L(p) = \dfrac{1}{2} p \cdot p$ with the dot denoting the ordinary inner product on $\mathbb{R}^3$. This equation can be interpreted as an extended free rigid body dynamics with an arbitrary, not necessarily positive-definite, real symmetric matrix as the inverse inertia tensor. The extended free rigid body dynamics is completely integrable with two first integrals H and L. Besides being a Lie-Poisson system, the extended free rigid body can be formulated as a Hamiltonian system on the cotangent bundle of a Lie group G_{H} whose Lie algebra is $\mathbb{R}^3_{\mathsf{H}}$. This formulation is a key to the geometric setting for the quantization problem. Since a Lie group whose Lie algebra is $\mathbb{R}^3_{\mathsf{H}}$ is not unique, the Lie group G_{H} to be chosen in this article will be listed in Sec. 3 because of necessity in quantum mechanics. The relation between two formulations will be mentioned briefly. It is to be noted that the Hamiltonian H in the ordinary sense is geometrized into the Lie algebra $\mathbb{R}^3_{\mathsf{H}}$ and then the function L, which is usually of geometric nature, plays the role of Hamiltonian. It should be emphasized further that, if two of the eigenvalues of the symmetric matrix H are equal, an additional $SO(2)$-symmetry emerges around a principal axis, like a symmetric top in the ordinary free rigid body dynamics. In this case, the solution to the equation of motion can be expressed in terms of elementary functions. Moreover, this additional symmetry will play an equivalent role in quantum mechanics.

In Sec.3, the quantization is performed for the extended free rigid body. In association with the classical two first integrals H and L, the left-invariant operators Δ_H and Δ_L on $L^2(G_{\mathsf{H}})$ are introduced. Since the element corresponding to Δ_H in the universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$ belongs to its centre, Δ_H and Δ_L mutually commute. The goal of the quantum dynamics for an extended free rigid body in this section is to solve the Schrödinger equation through the simultaneous spectral resolution of Δ_H and Δ_L. This problem amounts to decomposing the left-regular representation of G_{H} into irreducible representations and thereby to giving the spectral resolution of the operator Δ_L on each irreducible representation. The former process can be carried out by the Plancherel formula for G_{H}, but the latter is difficult to perform explicitly, except for the case that the extended rigid body is an extended symmetric top. For the extended free symmetric top, the spectral resolution of Δ_L on each irreducible representation of G_{H} can be given explicitly on account of the additional $SO(2)$-symmetry. It should be pointed out that the spectra presented in this paper generalizes those for the ordinary free symmetric top given in [10].

2. Classical Dynamics

2.1. Lie-Poisson Dynamics

After [4], the bracket $[\cdot,\cdot]_{\mathsf{H}}$ associated with a real symmetric 3×3 matrix H is defined on the three-dimensional Euclidean space $\mathbb{R}^3$ through

$$[u,v]_{\mathsf{H}} = \mathsf{H}(u\times v), \quad u,v\in\mathbb{R}^3, \tag{2.1}$$

where $\times$ denotes the standard exterior product on $\mathbb{R}^3$. Then, the following can be verified by a straightforward calculation.

Lemma 1. *The pair* $(\mathbb{R}^3,[\cdot,\cdot]_{\mathsf{H}})$ *is a three-dimensional Lie algebra for each real symmetric* 3×3 *matrix* H*, which is denoted by* $\mathbb{R}^3_{\mathsf{H}}$*.*

The types of the Lie algebra $\mathbb{R}^3_{\mathsf{H}}$ are classified according to the types of the signature of the symmetric matrix H, where the signature $\operatorname{sgn}(\mathsf{H})$ of a real symmetric matrix H is defined to be the triple (l,m,n) of the numbers of the positive, null, and negative eigenvalues of H, respectively.

Proposition 1. *The Lie algebra* $\mathbb{R}^3_{\mathsf{H}}$ *is isomorphic to one of the following:*

$$\mathbb{R}^3_{\mathsf{H}} \cong \begin{cases} \mathfrak{so}(3) & \textit{if } \operatorname{sgn}(\mathsf{H}) = (3,0,0),(0,0,3), \\ \mathfrak{so}(2,1) & \textit{if } \operatorname{sgn}(\mathsf{H}) = (2,0,1),(1,0,2), \\ \mathfrak{so}(2) \ltimes \mathbb{R}^2 & \textit{if } \operatorname{sgn}(\mathsf{H}) = (2,1,0),(0,1,2), \\ \mathfrak{so}(1,1) \ltimes \mathbb{R}^2 & \textit{if } \operatorname{sgn}(\mathsf{H}) = (1,1,1), \\ \mathfrak{h}_1 & \textit{if } \operatorname{sgn}(\mathsf{H}) = (1,2,0),(0,2,1), \\ \mathbb{R}^3 & \textit{if } \operatorname{sgn}(\mathsf{H}) = (0,3,0). \end{cases}$$

Here, the Lie algebra $\mathfrak{h}_1$ *is the Heisenberg algebra of dimension three.*

The proof can be performed by constructing isomorphisms in respective cases, which is straightforward.

On the dual space $\mathbb{R}^{3\,*}_{\mathsf{H}}$ of the Lie algebra $\mathbb{R}^3_{\mathsf{H}}$, the canonical Lie-Poisson structure $\{\cdot,\cdot\}_{\mathsf{H}}$ can be defined through

$$\begin{aligned} \{F,G\}_{\mathsf{H}}(p) &= \langle p, [(dF)_p, (dG)_p]_{\mathsf{H}} \rangle \\ &= p \cdot \mathsf{H}(\nabla F \times \nabla G), \end{aligned} \tag{2.2}$$

where $p \in \mathbb{R}^{3\,*}_{\mathsf{H}}$, and $F, G \in \mathcal{C}^\infty(\mathbb{R}^{3\,*}_{\mathsf{H}})$, and where $\langle\cdot,\cdot\rangle$ denotes the paring between $\mathbb{R}^3_{\mathsf{H}}$ and $\mathbb{R}^{3\,*}_{\mathsf{H}}$. We identify $\mathbb{R}^3_{\mathsf{H}}$ and $\mathbb{R}^{3\,*}_{\mathsf{H}}$ with their underlying vector space $\mathbb{R}^3$, so that the pairing $\langle\cdot,\cdot\rangle$ is, in fact, given by the inner product (dot product) $\cdot$ with respect to the ordinary Euclidean metric on $\mathbb{R}^3$. For any $F \in \mathcal{C}^\infty(\mathbb{R}^{3\,*}_{\mathsf{H}})$, we regard the differential $(dF)_p \in T^*_p(\mathbb{R}^{3\,*}_{\mathsf{H}})$, a linear form on $T_p(\mathbb{R}^{3\,*}_{\mathsf{H}})$, as an element of $T^*_0(\mathbb{R}^{3\,*}_{\mathsf{H}}) = \mathbb{R}^{3\,**}_{\mathsf{H}} = \mathbb{R}^3_{\mathsf{H}}$ through the parallel translation. We denote it by ∇F if we consider it to be a gradient vector field on $\mathbb{R}^3$. In Eq.(2.2), the gradient vector fields ∇F and ∇G are evaluated at p. We define the function H as $H(p) = \dfrac{1}{2} p \cdot \mathsf{H} p$, $p \in \mathbb{R}^{3\,*}_{\mathsf{H}}$. The following is easy to verify.

Lemma 2. *The pair* $(\mathbb{R}^{3\,*}_{\mathsf{H}}, \{\cdot,\cdot\}_{\mathsf{H}})$ *is a Poisson manifold and the function* H *is a Casimir with respect to the Poisson structure* $\{\cdot,\cdot\}_{\mathsf{H}}$*, i.e.,* $\{H,F\}_{\mathsf{H}} = 0$*, for any* $F \in \mathcal{C}^\infty(\mathbb{R}^{3\,*}_{\mathsf{H}})$.

For a function $G \in \mathcal{C}^\infty(\mathbb{R}^{3\,*}_{\mathsf{H}})$, the associated Hamiltonian vector field X_G with respect to the Poisson structure $\{\cdot,\cdot\}_{\mathsf{H}}$ is defined through

$$X_G(F) = \{G,F\}_{\mathsf{H}}, \quad F \in \mathcal{C}^\infty(\mathbb{R}^{3\,*}_{\mathsf{H}}), \tag{2.3}$$

which turns out to be expressed, in the vector form, as

$$X_G = \nabla H \times \nabla G. \tag{2.4}$$

Choosing the function $L(p) = \dfrac{1}{2} p \cdot p$, $p \in \mathbb{R}^{3\,*}_{\mathsf{H}}$, we define the extended free rigid body as follows:

Definition 1. *For any real symmetric 3×3 matrix H, the extended free rigid body associated with H is defined to be the Lie-Poisson system $({\mathbb{R}^3_{\mathsf{H}}}^*, \{\cdot,\cdot\}_{\mathsf{H}}, L)$ with the Hamiltonian L on the Poisson manifold $({\mathbb{R}^3_{\mathsf{H}}}^*, \{\cdot,\cdot\}_{\mathsf{H}})$, whose Hamilton equation is expressed, in the vector form, as*

$$\frac{dp}{dt} = \nabla H \times \nabla L = (\mathsf{H}p) \times p. \tag{2.5}$$

The time-derivative of a function F on ${\mathbb{R}^3_{\mathsf{H}}}^*$ is given by $\dfrac{dF}{dt} = X_L(F) = \{L, F\}_{\mathsf{H}}$. From this, it follows that the extended free rigid body dynamics has two first integrals H and L. The equation of motion (2.5) can be interpreted as an extension of the ordinary free rigid body equation with the inverse inertia tensor extended to an arbitrary real symmetric 3×3 matrix. We call this equation of motion the Euler equation as well. Since the function H is not bounded from below in general, it is no longer a kinetic energy function. The function L plays the role of the energy.

Remark 1. According to [7], the Lie-Poisson bracket in the ordinary sense may be defined in two ways, $\{F, G\}^{\pm}(p) = \pm p \cdot (\nabla F \times \nabla G)$, and the minus sign is chosen for the ordinary rigid body dynamics. We have chosen the plus sign for the definition of the Lie-Poisson bracket out of the possible signs $\pm$ for definitions $\{F, G\}^{\pm}_{\mathsf{H}} = \pm p \cdot (\nabla F \times \nabla G)$. Hence, the equation of motion with H positive-definite has the opposite sign to the ordinary Euler equation for a free rigid body.

Remark 2. The Euler equation for an extended free rigid body is bi-Hamiltonian with respect to the Lie-Poisson structure $\{\cdot,\cdot\}_{\mathsf{H}}$ and the standard Lie-Poisson structure $\{\cdot,\cdot\}_{\mathsf{E}}$, where E stands for the unit matrix. In fact, the equation of motion can be associated with the Lie-Poisson system $({\mathbb{R}^3_{\mathsf{E}}}^*, \{\cdot,\cdot\}_{\mathsf{E}}, -H)$ as well as $({\mathbb{R}^3_{\mathsf{H}}}^*, \{\cdot,\cdot\}_{\mathsf{H}}, L)$, and the brackets $\{\cdot,\cdot\}_{\mathsf{H}}$ and $\{\cdot,\cdot\}_{\mathsf{E}}$ are compatible in the sense that $a\{\cdot,\cdot\}_{\mathsf{E}} + b\{\cdot,\cdot\}_{\mathsf{H}} = \{\cdot,\cdot\}_{a\mathsf{E}+b\mathsf{H}}$, for any $a, b \in \mathbb{R}$. The bi-Hamiltonian structure of the Euler equation for a free rigid body is already generalized to that on $\mathfrak{so}(n)$ and $\mathfrak{sl}(n, \mathbb{R})$ and studied in [2, 3, 8].

2.2. Symplectic Stratification

Since much of the material in this subsection is well-known in geometric mechanics, we refer to [11, Chapter 6] for the proofs of several statements in what follows. Let G_{H} be a Lie group whose Lie algebra is $\mathbb{R}^3_{\mathsf{H}}$. The coadjoint action of G_{H} on the dual space ${\mathbb{R}^3_{\mathsf{H}}}^*$ of the Lie algebra $\mathbb{R}^3_{\mathsf{H}}$ is defined through

$$\langle \mathrm{Ad}^*_g\, p, u \rangle = \langle p, \mathrm{Ad}_{g^{-1}} u \rangle, \tag{2.6}$$

where $p \in \mathbb{R}^{3\,*}_{\mathsf{H}}$, $u \in \mathbb{R}^3_{\mathsf{H}}$, $g \in G_{\mathsf{H}}$, and where Ad stands for the adjoint action of G_{H}. The coadjoint action makes $\mathbb{R}^{3\,*}_{\mathsf{H}}$ stratified into coadjoint orbits:

$$\mathbb{R}^{3\,*}_{\mathsf{H}} = \bigsqcup_{\alpha \in A} \mathcal{S}_\alpha, \tag{2.7}$$

where $\mathcal{S}_\alpha$'s are the coadjoint orbits. Each coadjoint orbit possesses the canonical symplectic form (well-known as the Kirillov-Kostant-Souriau form and abbreviated to the KKS form as usual) ω_α as follows: On account of Eq.(2.6), the coadjoint action of $\mathbb{R}^3_{\mathsf{H}}$ on $\mathbb{R}^{3\,*}_{\mathsf{H}}$ is given by

$$\langle \mathrm{ad}^*_v\, p, u\rangle = -\langle p, \mathrm{ad}_v u\rangle, \tag{2.8}$$

where $p \in \mathbb{R}^{3\,*}_{\mathsf{H}}$, $u, v \in \mathbb{R}^3_{\mathsf{H}}$, so that $\mathrm{ad}^*_v p = -(\mathsf{H}p) \times v$ for $v \in \mathbb{R}^3_{\mathsf{H}}$. The ad^* generates the tangent space at p to the coadjoint orbit $\mathcal{S}_\alpha$ through p. The KKS form ω_α on $\mathcal{S}_\alpha$ is defined through

$$\omega_\alpha(\mathrm{ad}^*_u\, p, \mathrm{ad}^*_v\, p) = \langle p, \mathsf{H}(u \times v)\rangle, \tag{2.9}$$

where $u, v \in \mathbb{R}^3_{\mathsf{H}}$. This symplectic form determines the Poisson structure $\{\cdot,\cdot\}_\alpha$ on $\mathcal{S}_\alpha$, which is the same as the induced one on $\mathcal{S}_\alpha$ from $\{\cdot,\cdot\}_{\mathsf{H}}$;

$$\{F, G\}_{\mathsf{H}}\,|_{\mathcal{S}_\alpha} = \{F|_{\mathcal{S}_\alpha}, G|_{\mathcal{S}_\alpha}\}_\alpha\,, \tag{2.10}$$

where $F, G \in \mathcal{C}^\infty(\mathbb{R}^{3\,*}_{\mathsf{H}})$, so that Eq.(2.7) provides a symplectic stratification of $\mathbb{R}^{3\,*}_{\mathsf{H}}$.

2.3. Cotangent Bundle Formulation

We mention the cotangent bundle formulation for an extended free rigid body. Let L_a denotes the left action by $a \in G_{\mathsf{H}} : g \mapsto ag$ for $g \in G_{\mathsf{H}}$. The cotangent bundle T^*G_{H} of G_{H} can be identified with the product $G_{\mathsf{H}} \times \mathbb{R}^{3\,*}_{\mathsf{H}}$ through the left-trivialization:

$$T^*G_{\mathsf{H}} \ni (g, \alpha_g) \mapsto (g, L^*_g \alpha_g) \in G_{\mathsf{H}} \times \mathbb{R}^{3\,*}_{\mathsf{H}}. \tag{2.11}$$

The functions H and L on $\mathbb{R}^{3\,*}_{\mathsf{H}}$ can be extended to the left-invariant functions $\tilde{H}$ and $\tilde{L}$ on T^*G_{H} by $\tilde{H}(g, \alpha_g) = H(L^*_g \alpha_g)$ and $\tilde{L}(g, \alpha_g) = L(L^*_g \alpha_g)$, respectively. Let $\pi : T^*G_{\mathsf{H}} \to G_{\mathsf{H}}$ be the canonical projection. The canonical one-form θ on T^*G_{H} is given by

$$\theta(\tilde{X})|_{(g,\alpha_g)} = \alpha_g(\pi_* \tilde{X}) = \langle p, {L_{g^{-1}}}_* \pi_* \tilde{X}\rangle, \tag{2.12}$$

where $\tilde{X} \in T_{(g,\alpha_g)}(T^*G_{\mathsf{H}})$ and $p = L^*_g \alpha_g$, and where $\mathbb{R}^3_{\mathsf{H}}$ is identified with the tangent space to G_{H} at the identity of G_{H}. The canonical symplectic form $\Omega_{G_{\mathsf{H}}}$ on T^*G_{H} is defined to be $\Omega_{G_{\mathsf{H}}} = -d\theta$. Thus, we obtain the Hamiltonian system $(T^*G_{\mathsf{H}}, \Omega_{G_{\mathsf{H}}}, \tilde{L})$. The Hamiltonian vector field $\tilde{\Xi}_{\tilde{L}}$ is defined through $d\tilde{L} = \iota_{\tilde{\Xi}_{\tilde{L}}}\Omega_{G_{\mathsf{H}}}$, where ι denotes the interior product by a vector field. The tangent space $T_{(g,\alpha_g)}(T^*G_{\mathsf{H}})$ can be identified with $T_g G_{\mathsf{H}} \times {\mathbb{R}^3_{\mathsf{H}}}^*$, according to which $\tilde{X}, \tilde{\Xi}_{\tilde{L}} \in T_{(g,\alpha_g)}(T^*G_{\mathsf{H}})$ can be written as $\tilde{X} = (X, X')$, $\tilde{\Xi}_{\tilde{L}} = (\Xi_L, \Xi'_L)$, where $X, \Xi_L \in T_g G_{\mathsf{H}}$ and $X', \Xi'_L \in {\mathbb{R}^3_{\mathsf{H}}}^*$. Note that the tangent vectors X' and Ξ'_L on ${\mathbb{R}^3_{\mathsf{H}}}^*$ at the origin are regarded as elements of ${\mathbb{R}^3_{\mathsf{H}}}^*$ itself in the standard manner. We have the following proposition. For the proof, see [1, p.315, Proposition 4.4.1.].

Proposition 2. *The Hamiltonian vector field $\tilde{\Xi}_{\tilde{L}} = (\Xi_L, \Xi'_L)$ is expressed as*

$$\begin{aligned} \Xi_L &= L_{g*} dL \\ \Xi'_L &= -\mathrm{ad}^*_{dL} p = (\mathsf{H}p) \times p. \end{aligned} \tag{2.13}$$

where the differential dL on ${\mathbb{R}^3_{\mathsf{H}}}^*$ is viewed as an element of $\mathbb{R}^3_{\mathsf{H}}$.

This shows that the vector filed $\tilde{\Xi}_{\tilde{L}}$ on T^*G_{H} induces the Hamiltonian vector field X_L associated with L on the Lie-Poisson system $\left({\mathbb{R}^3_{\mathsf{H}}}^*, \{\cdot,\cdot\}_{\mathsf{H}}, L\right)$. Taking into account the momentum map $\mu : T^*G_{\mathsf{H}} \cong G_{\mathsf{H}} \times {\mathbb{R}^3_{\mathsf{H}}}^* \to {\mathbb{R}^3_{\mathsf{H}}}^*$ given by $(g, p) \mapsto \mathrm{Ad}^*_{g^{-1}} p$ with $(g, p) \in G_{\mathsf{H}} \times {\mathbb{R}^3_{\mathsf{H}}}^*$, we have the following commutative diagram:

$$\begin{array}{ccccc} \mu^{-1}(p) & \hookrightarrow & T^*G_{\mathsf{H}} & \xrightarrow{\tilde{L}} & \mathbb{R} \\ \downarrow & & \downarrow \mu & & \\ \mathcal{S}_\alpha & \hookrightarrow & {\mathbb{R}^3_{\mathsf{H}}}^* & \xrightarrow{L} & \mathbb{R} \end{array}$$

Here, the coadjoint orbit $\mathcal{S}_\alpha$ through $p \in {\mathbb{R}^3_{\mathsf{H}}}^*$ is diffeomorphic to the quotient of the momentum manifold $\mu^{-1}(p)$ by the action of the stabilizer $(G_{\mathsf{H}})_p \subset G_{\mathsf{H}}$ at $p \in {\mathbb{R}^3_{\mathsf{H}}}^*$. Since the Hamiltonian system $(T^*G_{\mathsf{H}}, \Omega_{G_{\mathsf{H}}}, \tilde{L})$ is left-invariant under the action of G_{H}, the reduction procedure can be applied to this dynamical system to provide the reduced Hamiltonian system that is isomorphic with $(\mathcal{S}_\alpha, \omega_\alpha, L|_{\mathcal{S}_\alpha})$.

2.4. The $SO(2)$-symmetry

In this subsection, we describe the extended free symmetric tops, which are the generalization of the ordinary free symmetric (or Lagrange) tops. Like the ordinary free symmetric top, the Euler equation for the extended free symmetric top can be integrated in terms of trigonometric functions. This is due to an additional $SO(2)$-symmetry.

Definition 2. *The extended free rigid body $({\mathbb{R}^3_{\mathsf{H}}}^*, \{\cdot,\cdot\}_{\mathsf{H}}, L)$ is called an extended free symmetric top, if two of the eigenvalues of the symmetric matrix H are equal.*

We take $\mathsf{H} = \mathrm{diag}(J_1, J_2, J_2)$ as the symmetric matrix H for an extended free symmetric top, without loss of generality. Then, the additional symmetry group $SO(2)$ acts on ${\mathbb{R}^3_{\mathsf{H}}}^*$ as follows:

$$\phi_\theta : {\mathbb{R}^3_{\mathsf{H}}}^* \ni p \mapsto e^{\theta \hat{\mathrm{e}}_1} p \in {\mathbb{R}^3_{\mathsf{H}}}^*, \quad \hat{\mathrm{e}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.14}$$

where $\theta \in \mathbb{R}$. Since

$$\frac{dp_1}{dt} = X_L(p_1) = \{L, p_1\}_{\mathsf{H}}(p) = (J_3 - J_2)p_2 p_3, \tag{2.15}$$

p_1 is a first integral for an extended free symmetric top with $J_2 = J_3$. Hence, the motion takes place on a plane perpendicular to the p_1-axis. The following proposition is obvious, since the integration of the Euler equation for an ordinary free symmetric top does not require the positive-definiteness of H.

Proposition 3. *The solution of the Euler equation for an extended free symmetric top* $({\mathbb{R}^3_{\mathsf{H}}}^*, \{\cdot,\cdot\}_{\mathsf{H}}, L)$ *with* $\mathsf{H} = \mathrm{diag}(J_1, J_2, J_2)$ *is given by*

$$p(t) = e^{t\omega \hat{\mathrm{e}}_1} c, \tag{2.16}$$

where $c = (c_1, c_2, c_3)^{\mathrm{T}}$ *and* $\omega = c_3(J_2 - J_1)$.

This proposition implies that the solution to the Euler equation for an extended free symmetric top is described as an orbit of the $SO(2)$-symmetry. In the next section, we will observe an analogous phenomenon in the quantum mechanics of the extended free symmetric top.

3. Quantum Dynamics

As was mentioned in the previous section, the extended free rigid body can be viewed as the Hamiltonian system $(T^*G_{\mathsf{H}}, \Omega_{G_{\mathsf{H}}}, \tilde{L})$. With this in mind, we set up a quantization of the extended free rigid body in the Schrödinger picture.

3.1. Quantization of an Extended Free Rigid Body

The classical extended free rigid body admits two Poisson-commuting first integrals H and L, which naturally induce two functions $\tilde{H}$ and $\tilde{L}$ on T^*G_{H}, respectively. These functions give rise to left-invariant symmetric bilinear forms on T^*G_{H}, which we denote by the same symbols for brevity. In particular, $\tilde{L}$ is a left-invariant Riemannian metric on G_{H}. Let $\nabla_{\tilde{L}}$ be the Levi-Civita connection of TG_{H} with respect to $\tilde{L}$, and ${\nabla_{\tilde{L}}}^*$ the connection of T^*G_{H} induced from $\nabla_{\tilde{L}}$. We introduce two operators Δ_H and Δ_L associated with the two first integrals H and L, respectively;

$$\Delta_H := -\mathrm{Trace}\ \tilde{H}({\nabla_{\tilde{L}}}^* \circ d), \tag{3.1}$$

$$\Delta_L := -\mathrm{Trace}\ \tilde{L}({\nabla_{\tilde{L}}}^* \circ d). \tag{3.2}$$

Here, we regard the exterior derivative d as the differential operator $d : \mathcal{C}^\infty(G_{\mathsf{H}}) \to \mathcal{C}^\infty(G_{\mathsf{H}}, T^*G_{\mathsf{H}})$, and the connection ${\nabla_{\tilde{L}}}^*$ as the covariant differential ${\nabla_{\tilde{L}}}^* : \mathcal{C}^\infty(G_{\mathsf{H}}, T^*G_{\mathsf{H}}) \to \mathcal{C}^\infty(G_{\mathsf{H}}, T^*G_{\mathsf{H}} \otimes T^*G_{\mathsf{H}})$. The bilinear forms $\tilde{H}$ and $\tilde{L}$ are thought of as bundle-homomorphisms by tensor multiplications from $T^*G_{\mathsf{H}} \otimes T^*G_{\mathsf{H}}$ to $TG_{\mathsf{H}} \otimes TG_{\mathsf{H}} \otimes T^*G_{\mathsf{H}} \otimes T^*G_{\mathsf{H}}$. The symbol “Trace” denotes the composition of contractions $\mathrm{Trace}_1^1 \circ \mathrm{Trace}_2^2$ of the $(2,2)$-tensors.

Let X_1, X_2, and X_3 be the generators of the Lie algebra $\mathbb{R}^3_{\mathsf{H}}$ such that

$$[X_i, X_j]_{\mathsf{H}} = \sum_{k=1}^{3} \epsilon_{ijk} J_k X_k, \qquad i, j, k = 1, 2, 3, \tag{3.3}$$

where ϵ_{ijk} is Eddington’s epsilon, and where J_k, $k = 1, 2, 3$, are the eigenvalues of the symmetric matrix H. We can assume that the generators X_i form an orthogonal basis with norms $\sqrt{2}$ with respect to the Euclidean metric L.

Proposition 4. *Two operators Δ_H and Δ_L can be identified with the elements*

$$-\frac{1}{2}(J_1 {X_1}^2 + J_2 {X_2}^2 + J_3 {X_3}^2) \tag{3.4}$$

and

$$-\frac{1}{2}({X_1}^2 + {X_2}^2 + {X_3}^2) \tag{3.5}$$

of the universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$, respectively.

Proof. Let θ^i be the dual basis to X_i with respect to $\tilde{L}$, $i = 1, 2, 3$. Regarding X_i and θ^i as left-invariant vector fields and as left-invariant differential forms on G_{H}, respectively, we can observe that the Christoffel symbols Γ^k_{ij} defined through $(\nabla_{\tilde{L}})_{X_i} X_j = \sum_{k=1}^{3} \Gamma^k_{ij} X_k$ satisfy $\Gamma^k_{ij} + \Gamma^j_{ik} = 0$ and $\Gamma^k_{ij} - \Gamma^k_{ji} = J_k \epsilon_{ijk}$. Particularly, $\Gamma^k_{ii} = 0$. Since the dual connection $\nabla^*_{\tilde{L}}$ is given by $(\nabla^*_{\tilde{L}})_{X_k} \theta^j = -\sum_{i=1}^{3} \Gamma^j_{ki} \theta^i$, we have, for any functions $f \in \mathcal{C}^\infty(G_{\mathsf{H}})$,

$$\nabla^*_{\tilde{L}} \circ df = \sum_{i,j=1}^{3} X_j X_i(f) \theta^j \otimes \theta^i - \sum_{i,j,k=1}^{3} X_i(f) \Gamma^i_{jk} \theta^j \otimes \theta^k, \tag{3.6}$$

so that

$$\mathrm{Trace}\ \tilde{H}(\nabla^*_{\tilde{L}} \circ df) = \frac{1}{2}\sum_{i,j=1}^{3}\left(J_i\delta_{ij}X_jX_i(f) - \sum_{k=1}^{3} J_j\delta_{jk}\Gamma^i_{jk}X_i(f)\right) = \frac{1}{2}\sum_{i=1}^{3} J_i {X_i}^2 f, \tag{3.7}$$

and

$$\mathrm{Trace}\ \tilde{L}(\nabla^*_{\tilde{L}} \circ df) = \frac{1}{2}\sum_{i,j=1}^{3}\left(\delta_{ij}X_jX_i(f) - \sum_{k=1}^{3} \delta_{jk}\Gamma^i_{jk}X_i(f)\right) = \frac{1}{2}\sum_{i=1}^{3} {X_i}^2 f. \tag{3.8}$$

This ends the proof.

Lemma 3. *The operator Δ_H is contained in the centre of the universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$.*

This can be proved straightforward.

The universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$ has a natural involution $D \mapsto D^*$ such that $(X + \sqrt{-1}Y)^* = -X + \sqrt{-1}Y$ for $X, Y \in \mathbb{R}^3_{\mathsf{H}}$. An element $D \in \mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$ is said to be Hermitian when $D^* = D$ holds. It is clear that the following is true.

Lemma 4. *The operators Δ_H and Δ_L are Hermitian.*

Now, we have to identify the Hilbert space on (a dense subset of) which Δ_H and Δ_L act. Let $d_{G_{\mathsf{H}}}$ be the left-invariant Haar measure on G_{H}. The space, $\mathcal{C}^\infty_0(G_{\mathsf{H}})$, of compactly supported functions is endowed with the inner product defined through

$$(f, g)_{G_{\mathsf{H}}} = \int_{G_{\mathsf{H}}} f(x)\overline{g(x)}d_{G_{\mathsf{H}}}(x), \tag{3.9}$$

where $f, g \in \mathcal{C}^\infty_0(G_{\mathsf{H}})$. We denote by $L^2(G_{\mathsf{H}})$ the completion of $\mathcal{C}^\infty_0(G_{\mathsf{H}})$ with respect to the inner product $(\cdot,\cdot)_{G_{\mathsf{H}}}$.

Proposition 5. *Δ_H and Δ_L are mutually commutative and essentially self-adjoint operators on $L^2(G_{\mathsf{H}})$.*

This is a consequence of a theorem by Nelson and Stinespring. See Warner's book [17, Chapter 4, §4.4.4] for details. Owing to Theorem 4.4.4.3 in p.268 of this book, Δ_L is essentially self-adjoint, and by Example (2) in p.269 of the same book, we can conclude that Δ_H, which is in the centre of the universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$, is essentially self-adjoint. Since the domains of Δ_H and Δ_L are not specified explicitly but are only known to be dense in $L^2(G_{\mathsf{H}})$, the commutativity of the two operators is rather ambiguous. However, the simultaneous spectral resolution exists for these operators, if the irreducible decomposition of the left-regular representation exists.

So far we have obtained the operators Δ_H and Δ_L on $L^2(G_{\mathsf{H}})$, which are associated with the first integrals of the classical extended free rigid body dynamics. Now, our interest centres on spectral resolution.

Quantization Problem: *Find the simultaneous spectral resolution of the two operators Δ_H and Δ_L on the Hilbert space $L^2(G_{\mathsf{H}})$.*

3.2. Plancherel Formula

A strategy for solving the quantization problem can be described as follows: Since the operator Δ_H is contained in the centre of the universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$, it acts by a constant multiplication of a real number on each irreducible representation space for the Lie group G_{H}. Then, the spectral resolution of Δ_H is available, if one can decompose the left-regular representation space $L^2(G_{\mathsf{H}})$ into irreducible representation spaces of G_{H}. At this stage, the geometric setting for our quantization problem is completed. The next task is to find the spectra of the "energy" operator Δ_L, which is performed through the simultaneous spectral resolution of the commutative operators Δ_H and Δ_L.

A key to the decomposition of $L^2(G_{\mathsf{H}})$ into irreducible unitary representations is the Plancherel formula. See, e.g., [17, Chapter 7] for details. With the assumption that the Lie group G_{H} is unimodular and of type I and satisfies the second axiom of countability, a theorem proved by I. E. Segal is put in the following form.

Theorem 1. *Let $\widehat{G_{\mathsf{H}}}$ denote the unitary dual of G_{H}, which is the set of the equivalence classes of irreducible unitary representations of G_{H} equipped with the Fell topology. (This topology is defined as follows: For a subset $S \subset \widehat{G_{\mathsf{H}}}$, the closure $\overline{S}$ is the set of the equivalence classes $[(\pi, V)] \in \widehat{G_{\mathsf{H}}}$ such that for any compact $K \subset G_{\mathsf{H}}$, $v \in V$, and $\epsilon > 0$, there exist an equivalence class $[(\pi', V')] \in S$ and an element $v' \in V'$ such that $\max_{x\in K}|(\pi(x)v, v) - (\pi'(x)v', v')| < \epsilon$.) Then, there exists a unique measure μ (called the Plancherel measure) on $\widehat{G_{\mathsf{H}}}$ such that*

$$||f||^2_{L^2(G_{\mathsf{H}})} = \int_{\widehat{G_{\mathsf{H}}}} ||\pi(f)||^2_{\mathrm{HS}} d\mu(\pi) \tag{3.10}$$

for any function $f \in L^2(G_{\mathsf{H}})$. Here, π ranges all the inequivalent irreducible unitary representations of G_{H}, and $||\pi(f)||_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of the multiplication operator $\pi(f)$ of the function f on G_{H} defined as

$$\pi(f) := \int_{G_{\mathsf{H}}} f(x)\pi(x) d_{G_{\mathsf{H}}}(x). \tag{3.11}$$

As a left-representation π of the Lie group G_{H} canonically induces a representation of the Lie algebra $\mathbb{R}^3_{\mathsf{H}}$ and, consequently, that of the universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$, we denote by the same letter π the derived representations after [16]. (See *loc. cit.* Chapter 0 for more details.) Hence, the notation $\pi(\Delta_H)$ makes sense, for example. Since the operator Δ_H acts on each irreducible unitary representation (π, V) as a constant multiplication; $\pi(\Delta_H) = \lambda_\pi \mathrm{id}_V$, the Plancherel formula gives rise to the spectral resolution of Δ_H;

$$\Delta_H = \int_{\widehat{G_{\mathsf{H}}}} d\mu(\pi)\lambda_\pi P(\pi), \tag{3.12}$$

where $P(\pi)$ denotes the projection to the irreducible representation π defined through

$$(f, P(\pi)g)_{G_{\mathsf{H}}} = \mathrm{Trace}(\pi(f)^*\pi(g)). \tag{3.13}$$

Finding the spectral resolution of the operator Δ_L on π results in the simultaneous spectral resolution of the two operators of $L^2(G_{\mathsf{H}})$.

Remark 3. From the viewpoint of analysis, it is inadequate to call Eq.(3.12) the spectral resolution of Δ_H, since P is not a partition of unity on $\mathbb{R}$. However, if one can evaluate Δ_H as λ_π on each irreducible representation, one can get the spectral resolution of Δ_H with a suitable spectral measure E in the form

$$\Delta_H = \int_{\mathbb{R}} \lambda dE(\lambda), \tag{3.14}$$

though the spectral measure cannot be determined explicitly.

3.3. Spectral Resolution of Δ_H and Δ_L

So far we have explained our strategy for the simultaneous spectral resolution of the two operators Δ_H and Δ_L in a rather abstract manner. Now, we need to obtain the explicit expression of the Plancherel formula for G_{H} and the spectral resolution of Δ_L on each irreducible representation of G_{H}. To this end, we assume that the Lie group G_{H} should be one of the following:

$$G_{\mathsf{H}} \cong \begin{cases} SO(3) & \text{if } \mathrm{sgn}(\mathsf{H}) = (3,0,0), (0,0,3), \\ SO_{\mathrm{e}}(2,1) & \text{if } \mathrm{sgn}(\mathsf{H}) = (2,0,1), (1,0,2), \\ SO(2) \ltimes \mathbb{R}^2 & \text{if } \mathrm{sgn}(\mathsf{H}) = (2,1,0), (0,1,2), \\ SO_{\mathrm{e}}(1,1) \ltimes \mathbb{R}^2 & \text{if } \mathrm{sgn}(\mathsf{H}) = (1,1,1), \\ \mathbb{H}^1 & \text{if } \mathrm{sgn}(\mathsf{H}) = (1,2,0), (0,2,1), \\ \mathbb{R}^3 & \text{if } \mathrm{sgn}(\mathsf{H}) = (0,3,0). \end{cases} \tag{3.15}$$

Here, $SO_{\mathrm{e}}(2,1)$ and $SO_{\mathrm{e}}(1,1)$ are the respective identity components of the Lorentz groups and $\mathbb{H}^1$ is the Heisenberg group of dimension three. These groups can be realized as 3×3 matrix groups and they are connected. One can check that these groups are unimodular and of type I, as is shown below. Hence, the Segal's theorem applies.

Lemma 5. *The Lie groups listed in* (3.15) *are unimodular and of type I.*

Proof. Since the modular function Δ of the Lie group G_{H} satisfies

$$\Delta(\exp X) = \exp(\mathrm{Trace}\ \mathrm{ad}_X) \tag{3.16}$$

for any element $X \in \mathbb{R}^3_{\mathsf{H}}$ (cf. [16, p.39]), it suffices for G_{H} to be unimodular to show that Trace $\mathrm{ad}_X = 0$ for any $X \in \mathbb{R}^3_{\mathsf{H}}$. For $X = \sum_{i=1}^3 x_i X_i$ and $Y = \sum_{i=1}^3 y_i X_i$, the relations $\mathrm{ad}_{X_i} X_j = [X_i, X_j]_{\mathsf{H}} = \displaystyle\sum_{k=1}^3 \epsilon_{ijk} J_k X_k$ give rise to

$$\mathrm{ad}_X Y = \sum_{i,j,k=1}^3 x_i y_j \epsilon_{ijk} J_k X_k. \tag{3.17}$$

Then,

$$\mathrm{Trace}\ \mathrm{ad}_X = \sum_{i=1}^3 \langle X_i, \mathrm{ad}_X X_i \rangle = 2 \sum_{i,j,k,l=1}^3 \delta_{ik}\delta_{ij} x_l \epsilon_{ljk} J_k = 0. \tag{3.18}$$

Further, it is known that all of commutative groups, connected semi-simple Lie groups, and connected solvable Lie groups whose exponential mapping is surjective are of type I, so that the groups listed in (3.15) are of type I.

Now, we proceed to exhibit the Plancherel formulae and the spectral resolution of the operator Δ_H according to the type of $\mathrm{sgn}(\mathsf{H})$ in an explicit manner. Successively, we look into structure of the irreducible unitary representations contributing to the Plancherel formulae, seeking for the simultaneous spectral resolution for Δ_L. Since the essential part of the following can be verified by using the results from representation theory together with some calculation, we omit details. See, e.g., [16] for more details.

We introduce the two elements

$$E_{\pm} = \sqrt{J_2} X_2 \mp \sqrt{-J_3} X_3 \tag{3.19}$$

in the complexification $\mathbb{R}^3_{\mathsf{H}} \otimes \mathbb{C}$ provided that $J_2 J_3 \neq 0$, and use them, in the same notation, as operators acting on $L^2(G_{\mathsf{H}})$. These operators are standard tools in the study of the representations of the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ (cf. [16]). The commutation relations among X_1 and $E_\pm$ are expressed as

$$[X_1, E_\pm]_{\mathsf{H}} = \pm\sqrt{-J_2J_3}E_\pm,$$
$$[E_+, E_-]_{\mathsf{H}} = 2J_1\sqrt{-J_2J_3}X_1. \tag{3.20}$$

Note that the operators Δ_H and Δ_L can be written in terms of X_1 and $E_\pm$ as

$$\Delta_H = -\frac{1}{2}\left(J_1X_1^2 + \frac{1}{2}\left(E_+E_- + E_-E_+\right)\right), \tag{3.21}$$

and

$$\Delta_L = -\frac{1}{8}\left(\frac{1}{J_2} - \frac{1}{J_3}\right)E_+^2 - \left\{\frac{1}{2}X_1^2 + \frac{1}{8}\left(\frac{1}{J_2} + \frac{1}{J_3}\right)(E_+E_- + E_-E_+)\right\}$$
$$-\frac{1}{8}\left(\frac{1}{J_2} - \frac{1}{J_3}\right)E_-^2, \tag{3.22}$$

respectively. These formulae can be derived through a straightforward calculation.

Although it is very difficult to obtain simultaneous spectral resolutions of Δ_H and Δ_L in an explicit manner, the problem can be solved explicitly, provided that the extended rigid body is an extended symmetric top, *i.e.*, if, say, $J_2 = J_3$, as will be seen in the following. The reason for this can be explained as follows: Like Eq.(2.15), an easy calculation shows that

$$[X_1, \Delta_L]_{\mathsf{H}} = \frac{J_2 - J_3}{2}(X_2X_3 + X_3X_2). \tag{3.23}$$

Thus, the operator X_1 commutes with Δ_L, if $J_2 = J_3$. In other words, the extended free symmetric top admits the additional symmetry $SO(2)$ generated by X_1. For this reason, the spectral resolution of Δ_L take a simpler form; the operators X_1 and Δ_L bear the same eigenspace.

In what follows, we describe the spectral resolution of Δ_H and Δ_L in an explicit manner for Lie groups G_{H} listed in (3.15).

1. $\mathrm{sgn}(\mathsf{H}) = (3,0,0)$ or $(0,0,3)$:
   In this case, one has $G_{\mathsf{H}} \cong SO(3)$, which is compact, so that the Peter-Weyl theorem applies. For any $f \in L^2(G_{\mathsf{H}})$, we have the Parseval-Plancherel formula

$$||f||^2_{L^2(SO(3))} = \sum_{\pi\in\widehat{SO(3)}} \dim(\pi)||\pi(f)||^2_{HS}$$
$$= \sum_{m=0}^{\infty}(2m+1)||\pi_{2m+1}(f)||^2_{HS}, \tag{3.24}$$

where we have used the fact that the irreducible unitary representations of $SO(3)$ are of odd dimension and unique up to equivalence classes in each dimension, and denoted the $(2m+1)$-dimensional representation by π_{2m+1}. From this formula together with the fact that $\pi_{2m+1}(\Delta_H) = \dfrac{\det(\mathsf{H})}{2} m(m+1)$, we get the spectral resolution of the operator Δ_H,

$$\Delta_H = \frac{\det(\mathsf{H})}{2} \sum_{m=0}^{\infty} m(m+1) E(2m+1), \tag{3.25}$$

where $\{E(2m+1)\}_{m=0}^{\infty}$ is the spectral measure.

The irreducible unitary representation (π_{2m+1}, V) can be decomposed into the direct sum of one-dimensional subspaces $V = \bigoplus_{j=-m}^{m} V_j$, where V_j's are the eigenspaces of X_1. We can choose a basis of V such that $v_j \in V_j$ and

$$\begin{aligned}
\pi_{2m+1}(X_1) v_j &= \sqrt{-J_2 J_3}\, j v_j, \\
\pi_{2m+1}(E_+) v_j &= \sqrt{-\det(\mathsf{H})}\sqrt{(m-j)(m+j+1)}\, v_{j+1}, \\
\pi_{2m+1}(E_-) v_j &= \sqrt{-\det(\mathsf{H})}\sqrt{(m+1-j)(m+j)}\, v_{j-1}.
\end{aligned} \tag{3.26}$$

From (3.22) and (3.26), the action of the operator Δ_L on v_j is expressed as

$$\begin{aligned}
&\pi_{2m+1}(\Delta_L) v_j \\
&= -\frac{\det(\mathsf{H})}{8} \Bigg[-\left(\frac{1}{J_2} - \frac{1}{J_3}\right) \sqrt{(m+j)(m+1-j)(m+j+1)(m-j)}\, v_{j+2} \\
&\quad + \left\{ -\frac{4}{J_1} j^2 + 2\left(\frac{1}{J_2} + \frac{1}{J_3}\right)\left(j^2 - m^2 - m\right) \right\} v_j \\
&\quad - \left(\frac{1}{J_2} - \frac{1}{J_3}\right) \sqrt{(m+1-j)(m+j)(m+2-j)(m+j-1)}\, v_{j-2} \Bigg].
\end{aligned} \tag{3.27}$$

If the matrix expression of Δ_L is diagonalized, the eigenvalue decomposition can be obtained. Though it is difficult to work out in an explicit manner for general $m \in \mathbb{N}$, if the extended free rigid body is an extended free symmetric top, *i.e.*, if $J_2 = J_3$, the representation matrix of Δ_L becomes diagonal, and the subspaces V_j are eigenspaces of Δ_L associated with the eigenvalue

$$\frac{\det(\mathsf{H})}{2} \left(\frac{1}{J_2} m(m+1) + \left(\frac{1}{J_1} - \frac{1}{J_2}\right) j^2 \right). \tag{3.28}$$

It is to be noted that these spectra are the same, within the constant factor, as given in [10], which are found by solving the Schrödinger equation for a symmetric top.

2. $\mathrm{sgn}(\mathsf{H}) = (2,0,1)$ or $(1,0,2)$:
In this case, one has $G_{\mathsf{H}} \cong SO_{\mathrm{e}}(2,1)$. Since $SO_{\mathrm{e}}(2,1)$ is isomorphic to $SL(2,\mathbb{R})/\{\pm 1\}$, we can use the Plancherel formula for $SL(2,\mathbb{R})$ (cf. [16, Chapter 8, §4] and [6, Chapter VIII, §4]). All the functions on $SL(2,\mathbb{R})/\{\pm 1\}$ are realized as those on $SL(2,\mathbb{R})$ that are invariant under the $\{\pm 1\}$-action. Since the second principal series of the representations of $SL(2,\mathbb{R})$ are not invariant under the action of the group $\{\pm 1\}$, no irreducible representation of that kind is needed in our Plancherel formula. The representations which appear in our case have only even eigensubspaces for the maximal compact subgroup of $SO_{\mathrm{e}}(2,1)$. Thus, the Plancherel formula for $SL(2,\mathbb{R})/\{\pm 1\}$ takes the form

$$||f||^2_{L^2(SO_{\mathrm{e}}(2,1))} = \int_0^\infty ||\pi^e_{\sqrt{-1}s}(f)||^2_{HS}\, s\tanh(\pi s) ds + \sum_{m=1}^\infty \frac{2m-1}{2}\left\{||\pi^+_{2m}(f)||^2_{HS} + ||\pi^-_{-2m}(f)||^2_{HS}\right\}, \tag{3.29}$$

where $\pi^e_{\sqrt{-1}s}$ is the first principal series with the parameter $s > 0$, and where $\pi^\pm_{\pm 2m}$ with $m \in \mathbb{N}$ are the holomorphic and the anti-holomorphic principal series, respectively. On account of

$$\pi^e_{\sqrt{-1}s}(\Delta_H) = -\frac{\det(\mathsf{H})}{8}\left(1+s^2\right) \tag{3.30}$$

and

$$\pi^\pm_{\pm 2m}(\Delta_H) = -\frac{\det(\mathsf{H})}{8}\left(1-(2m-1)^2\right), \tag{3.31}$$

the spectral resolution of the operator Δ_H takes the form

$$\Delta_H = -\frac{\det(\mathsf{H})}{8}\left\{\int_0^\infty (1+s^2)dE(s) + \sum_{m=1}^\infty \{1-(2m-1)^2\}\tilde{E}(m)\right\}, \tag{3.32}$$

where $\left\{E(s)\right\}_{s\in(0,\infty)} \cup \left\{\tilde{E}(m)\right\}_{m=1}^\infty$ is the spectral measure.

Now, we look into the structure of the irreducible representations appearing in the Plancherel formula. We assume that $J_1 < 0 < J_2, J_3$, for simplicity.

(a) The first principal series $(\pi^e_{\sqrt{-1}s}, V)$ with $s > 0$.
The representation space V can be decomposed into the direct sum of one-dimensional subspaces: $V = \bigoplus_{j=-\infty}^{\infty} V_{2j}$, where V_{2j} are eigenspaces of

X_1. We can choose a basis of V such that $v_{2j} \in V_{2j}$ and

$$\begin{aligned}
\pi^e_{\sqrt{-1}s}(X_1)v_{2j} &= \sqrt{-J_2 J_3}\, j v_{2j},\\
\pi^e_{\sqrt{-1}s}(E_+)v_{2j} &= \sqrt{\det(\mathsf{H})}\,\alpha_{2j} v_{2(j+1)},\\
\pi^e_{\sqrt{-1}s}(E_-)v_{2j} &= \sqrt{\det(\mathsf{H})}\,\overline{\alpha_{2(j-1)}}\, v_{2(j-1)},
\end{aligned} \tag{3.33}$$

where the coefficients are defined as $\alpha_{2j} = \dfrac{2j+1+\sqrt{-1}s}{2}$ for $j \in \mathbb{Z}$. The operator Δ_H is then evaluated on these representations as in (3.30). The operator Δ_L is represented, with respect to the basis above, as

$$\begin{aligned}
\pi^e_{\sqrt{-1}s}(\Delta_L)v_{2j} = -\frac{\det(\mathsf{H})}{8}\Bigg[&\alpha_{2j}\alpha_{2(j+1)}\left(\frac{1}{J_2}-\frac{1}{J_3}\right)v_{2(j+2)}\\
&+\left\{-4\frac{j^2}{J_1}+\left(\frac{1}{J_2}+\frac{1}{J_3}\right)\left(|\alpha_{2(j-1)}|^2+|\alpha_{2j}|^2\right)\right\}v_{2j}\\
&+\overline{\alpha_{2(j-1)}\alpha_{2(j-2)}}\left(\frac{1}{J_2}-\frac{1}{J_3}\right)v_{2(j-2)}\Bigg].
\end{aligned} \tag{3.34}$$

If $J_2 = J_3$, the above representation of Δ_L becomes simple and the subspaces V_j are eigenspaces of Δ_L associated with the eigenvalue

$$-\frac{\det(\mathsf{H})}{8}\left\{-4\frac{j^2}{J_1}+\frac{1}{J_2}\left(4j^2+1+s^2\right)\right\}. \tag{3.35}$$

(b) The holomorphic principal series (π^+_{2m}, V) with $m \in \mathbb{N}$.
The representation space V can be decomposed into the direct sum of one-dimensional subspaces: $V = \bigoplus_{j=0}^{\infty} V_{2(m+j)}$, where $V_{2(m+j)}$ are the eigenspaces of X_1. We can choose a basis of V such that $v_{2(m+j)} \in V_{2(m+j)}$ and

$$\begin{aligned}
\pi^+_{2m}(X_1)v_{2(m+j)} &= \sqrt{-J_2 J_3}\,(m+j)v_{2(m+j)},\\
\pi^+_{2m}(E_+)v_{2(m+j)} &= \sqrt{\det(\mathsf{H})}\,\beta_{2(m+j)} v_{2(m+j+1)},\\
\pi^+_{2m}(E_-)v_{2(m+j)} &= \sqrt{\det(\mathsf{H})}\,\overline{\beta_{2(m+j-1)}}\, v_{2(m+j-1)},
\end{aligned} \tag{3.36}$$

where the coefficients are chosen so that

$$|\beta_{2(m+j)}| = \frac{1}{2}\sqrt{\{2(m+j)+1\}^2-(2m-1)^2}. \tag{3.37}$$

The operator Δ_H is evaluated on these representations as in (3.31). The representation of the operator Δ_L with respect to the basis above is given by

$$\begin{aligned}
&\pi^+_{2m}(\Delta_L)v_{2(m+j)} \\
&= -\frac{\det(\mathsf{H})}{8}\Bigg[\beta_{2(m+j)}\beta_{2(m+j+1)}\left(\frac{1}{J_2}-\frac{1}{J_3}\right)v_{2(m+j+2)} \\
&\quad+\left\{-4\frac{(m+j)^2}{J_1}+\left(\frac{1}{J_2}+\frac{1}{J_3}\right)\left(|\beta_{2(m+j-1)}|^2+|\beta_{2(m+j)}|^2\right)\right\}v_{2(m+j)} \\
&\quad+\overline{\beta_{2(m+j-1)}\beta_{2(m+j-2)}}\left(\frac{1}{J_2}-\frac{1}{J_3}\right)v_{2(m+j-2)}\Bigg]. \qquad (3.38)
\end{aligned}$$

If $J_2 = J_3$, the above representation matrix of Δ_L becomes diagonal and the subspaces V_j are eigenspaces of Δ_L with the eigenvalue

$$-\frac{\det(\mathsf{H})}{8}\left\{-4\frac{(m+j)^2}{J_1}+\frac{1}{J_2}\left(4(m+j)^2+1-(2m-1)^2\right)\right\}. \qquad (3.39)$$

(c) The anti-holomorphic principal series (π^-_{-2m}, V) with $m \in \mathbb{N}$.
The representation space V can be decomposed into the direct sum of one-dimensional subspaces: $V = \bigoplus_{j=0}^{\infty} V_{-2(m+j)}$. We can choose a basis of V such that $v_{-2(m+j)} \in V_{-2(m+j)}$ and

$$\begin{aligned}
\pi^-_{-2m}(X_1)v_{-2(m+j)} &= -\sqrt{-J_2J_3}(m+j)v_{-2(m+j)}, \\
\pi^-_{-2m}(E_+)v_{-2(m+j)} &= \sqrt{\det(\mathsf{H})}\gamma_{-2(m+j)}v_{-2(m+j-1)}, \\
\pi^-_{-2m}(E_-)v_{-2(m+j)} &= \sqrt{\det(\mathsf{H})}\overline{\gamma_{-2(m+j+1)}}v_{-2(m+j+1)}, \qquad (3.40)
\end{aligned}$$

where the coefficients are chosen so that

$$|\gamma_{-2(m+j)}| = \frac{1}{2}\sqrt{(2(m+j)-1)^2-(2m+1)^2}. \qquad (3.41)$$

The operator Δ_H is evaluated on these representations as in (3.31). The representation of the operator Δ_L with respect to the basis above is of the form,

$$\begin{aligned}
&\pi^-_{-2m}(\Delta_L)v_{-2(m+j)} \\
&= -\frac{\det(\mathsf{H})}{8}\Bigg[\gamma_{-2(m+j)}\gamma_{-2(m+j-1)}\left(\frac{1}{J_2}-\frac{1}{J_3}\right)v_{-2(m+j-2)} \\
&\quad + \left\{-4\frac{(m+j)^2}{J_1}+\left(\frac{1}{J_2}+\frac{1}{J_3}\right)\left(|\gamma_{-2(m+j-1)}|^2+|\gamma_{-2(m+j)}|^2\right)\right\}v_{-2(m+j)} \\
&\quad + \overline{\gamma_{-2(m+j+1)}\gamma_{-2(m+j+2)}}\left(\frac{1}{J_2}-\frac{1}{J_3}\right)v_{-2(m+j+2)}\Bigg]. \qquad (3.42)
\end{aligned}$$

If $J_2 = J_3$, the above representation of Δ_L becomes diagonal, and the subspaces V_j are eigenspaces of Δ_L associated with the eigenvalue

$$-\frac{\det(\mathsf{H})}{8}\left\{-4\frac{(m+j)^2}{J_1}+\frac{1}{J_2}\left(4(m+j)^2+1-(2m-1)^2\right)\right\}. \qquad (3.43)$$

3. $\mathrm{sgn}(\mathsf{H}) = (2,1,0)$ or $(0,1,2)$:
In this case, G_{H} is isomorphic to the Euclidean motion group $SE(2) := SO(2) \ltimes \mathbb{R}^2$, whose Plancherel formula can be written as

$$||f||^2_{L^2(SE(2))} = \int_0^\infty ||U_r(f)||^2_{HS}\, r dr \qquad (3.44)$$

for any function $f \in L^2(SE(2))$ (see [15, Chapter IV, §4]). Here, $U_r = \mathrm{Ind}^{SE(2)}_{\mathbb{R}^2,\alpha}$ is a unique irreducible representation for $r \in (0,\infty)$ given by the method of induced representations (see [16, Chapter 5, §4]), where $\alpha \in \mathbb{R}^2$ is a vector with norm r and the representation is unique up to equivalence provided that $|\alpha| = r$. We assume that $J_1 = 0 < J_2, J_3$, for simplicity. The spectral resolution of the operator Δ_H is given by

$$\Delta_H = \frac{J_2J_3}{2}\int_0^\infty r^2 dE(r), \qquad (3.45)$$

where $\{E(r)\}_{r\in[0,\infty)}$ is the spectral measure.

The irreducible unitary representation (U_r, V) with $U_r = \mathrm{Ind}^{SE(2)}_{\mathbb{R}^2,\alpha}$ admits the decomposition, $V = \bigoplus_{j=-\infty}^{\infty} V_j$, such that there is a basis $v_j \in V_j$ satisfying

$$\begin{aligned}
U_r(X_1)v_j &= \sqrt{-J_2J_3}\, j v_j, \\
U_r(E_\pm)v_j &= \sqrt{-J_2J_3}\, r v_{j\pm1}. \qquad (3.46)
\end{aligned}$$

The operator Δ_H is evaluated as $U_r(\Delta_H) = \dfrac{J_2 J_3}{2} r^2$ on this irreducible representation. The representation of the operator Δ_L with respect to the above basis is described as

$$U_r(\Delta_L) v_j = \frac{J_2 J_3}{8} \Bigg[\left(\frac{1}{J_2} - \frac{1}{J_3} \right) r^2 v_{j+2} + \left\{ 4j^2 + 2 \left(\frac{1}{J_2} + \frac{1}{J_3} \right) r^2 \right\} v_j + \left(\frac{1}{J_2} - \frac{1}{J_3} \right) r^2 v_{j-2} \Bigg]. \tag{3.47}$$

If $J_2 = J_3$, the above representation of Δ_L has the invariant subspaces V_j, which are eigenspaces of Δ_L associated with the eigenvalue

$$\frac{J_2^2}{2} \left(j^2 + \frac{r^2}{J_2} \right). \tag{3.48}$$

4. $\mathrm{sgn}(\mathsf{H}) = (1, 1, 1)$:
In this case, G_{H} is isomorphic to the Poincaré group $\mathcal{L}_{\mathrm{e}}(2) := SO_{\mathrm{e}}(1,1) \ltimes \mathbb{R}^2$ whose Plancherel formula can be written as

$$||f||^2_{L^2(\mathcal{L}_{\mathrm{e}}(2))} = 2 \int_{-\infty}^{\infty} ||U_\lambda(f)||^2_{HS}\, d\lambda \tag{3.49}$$

for any function $f \in L^2(\mathcal{L}_{\mathrm{e}}(2))$. Here, $U_\lambda = \mathrm{Ind}_{\mathbb{R}^2,\alpha}^{\mathcal{L}_{\mathrm{e}}(2)}$ with $\lambda \in (-\infty, \infty)$ is a unique irreducible representation given by the method of induced representations (cf. [16, Chapter 5, §4]). The parameter $\alpha = (p, q) \in \mathbb{R}^2$ in the symbol of the induced representation is a vector with $pq = \lambda$ and the representation is unique up to equivalence provided that $pq = \lambda$. See the Appendix for the proof of the Plancherel formula in this case. We assume that $J_1 = 0$ and $J_2 > 0 > J_3$, for simplicity. The spectral resolution of the operator Δ_H is given by

$$\Delta_H = \frac{J_2 J_3}{2} \int_{-\infty}^{\infty} \lambda^2 dE(\lambda) = J_2 J_3 \int_0^{\infty} \lambda^2 dE(\lambda), \tag{3.50}$$

where $\{E(\lambda)\}_{\lambda \in (0,\infty)}$ is the spectral measure.

The irreducible unitary representation $U_\lambda = \mathrm{Ind}_{\mathbb{R}^2,\alpha}^{\mathcal{L}_{\mathrm{e}}(2)}$ can be realized on the Hilbert space $\mathcal{H}_{\phi_\alpha}$ defined through (A.3) and (A.4) in the Appendix. The generators X_1, X_2, and X_3 acts on $f \in \mathcal{H}_{\phi_\alpha}$ as

$$\begin{aligned} U_\lambda(X_1) f &= \sqrt{-J_2 J_3} \frac{\partial f}{\partial t}, \\ U_\lambda(X_2) f &= \frac{\sqrt{-J_3}}{2} \left(p e^t + q e^{-t} \right) f, \\ U_\lambda(X_3) f &= \frac{\sqrt{J_2}}{2} \left(p e^t - q e^{-t} \right) f, \end{aligned} \tag{3.51}$$

where $t \in \mathbb{R}$ denotes the parameter of the one-parameter subgroup of $\mathcal{L}_e(2)$ generated by X_1. The realization of the generators X_1, X_2, and X_3 is given by (A.7) in the Appendix. The operator Δ_H can be evaluated as $U_\lambda(\Delta_H) = \dfrac{J_2J_3}{2}\lambda^2$ on this irreducible representation. The operator Δ_L acts on $f \in \mathcal{H}_{\phi_\alpha}$ by

$$U_\lambda(\Delta_L)f = \frac{J_2J_3}{2}\left[\frac{\partial^2 f}{\partial t^2} + \left\{\left(\frac{1}{J_2} - \frac{1}{J_3}\right)\left(p^2e^{2t} + q^2e^{-2t}\right) + 2\left(\frac{1}{J_2} + \frac{1}{J_3}\right)pq\right\}f\right]. \tag{3.52}$$

Since $J_1 = 0$, and $J_2 > 0 > J_3$, no extended symmetric top exists for the Poincaré group $\mathcal{L}_e(2)$.

5. $\mathrm{sgn}(\mathsf{H}) = (1, 2, 0)$ or $(0, 2, 1)$:
In this case, G_{H} is isomorphic to the Heisenberg group $\mathbb{H}^1$ of dimension three. See [16, Chapter 1] for more details on $\mathbb{H}^1$. The Plancherel formula for $\mathbb{H}^1$ is written as

$$||f||^2_{L^2(\mathbb{H}^1)} = \int_{-\infty}^{\infty} ||\pi_\lambda(f)||^2_{HS}|\lambda|d\lambda, \tag{3.53}$$

where π_λ is the irreducible unitary representation with the parameter $\lambda \in \mathbb{R}$. We assume that $J_1 > 0 = J_2 = J_3$, for simplicity. The spectral resolution of the operator Δ_H can be written as

$$\Delta_H = \frac{J_1}{2}\int_{-\infty}^{\infty} \lambda^2 dE(\lambda), \tag{3.54}$$

where $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is the spectral measure.

To gain an insight into the structure of the irreducible unitary representation of G_{H} in this case, we introduce two elements $E'_\pm := X_2 \mp \sqrt{-1}X_3$ in the universal enveloping algebra $\mathfrak{U}(\mathbb{R}^3_{\mathsf{H}})$. The commutation relations among X_1 and $E'_\pm$ are then expressed as

$$\begin{aligned}&[X_1, E'_\pm]_{\mathsf{H}} = 0,\\ &[E'_+, E'_-]_{\mathsf{H}} = 2\sqrt{-1}J_1X_1.\end{aligned} \tag{3.55}$$

The irreducible representation (π_λ, V) admits the decomposition of V into the direct sum of one-dimensional subspaces: $V = \displaystyle\bigoplus_{j=0}^{\infty} V_j$, according to which we can take the basis $v_j \in V_j$ satisfying

$$\begin{aligned}\pi_\lambda(X_1)v_j &= \sqrt{-1}\lambda v_j,\\ \pi_\lambda(E'_+)v_j &= \sqrt{2\lambda J_1}\sqrt{j+1}v_{j+1},\\ \pi_\lambda(E'_-)v_j &= \sqrt{2\lambda J_1}\sqrt{j}v_{j-1}.\end{aligned} \tag{3.56}$$

The operator Δ_H can be evaluated as $\pi_\lambda(\Delta_H) = \dfrac{J_1}{2}\lambda^2$ on this irreducible representation. In terms of X_1' and $E_\pm'$, the operator Δ_L is expressed as

$$\Delta_L = -\frac{1}{2}{X_1}^2 - \frac{1}{4}\left(E_+'E_-' + E_-'E_+'\right), \tag{3.57}$$

and represented with respect to the basis v_j as

$$\pi_\lambda(\Delta_L)v_j = -\frac{\lambda}{2}\left\{J_1(2j+1) - \lambda\right\}v_j. \tag{3.58}$$

6. $\mathrm{sgn}(\mathsf{H}) = (0,3,0)$:
In this case, the Lie group G_{H} is isomorphic to the ordinary Abelian group $\mathbb{R}^3$ whose Plancherel formula is actually that of the ordinary Fourier transform,

$$||f||^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 d\xi, \tag{3.59}$$

where $\hat{f}(\xi) = \dfrac{1}{(2\pi)^{3/2}}\displaystyle\int_{\mathbb{R}^3} f(x)e^{-\sqrt{-1}\xi x}dx$ is the Fourier transform of f. The operator Δ_H is trivially zero in this case:

$$\Delta_H = 0. \tag{3.60}$$

The spectral resolution of the operator Δ_L is

$$\Delta_L = \int_0^\infty \lambda dE(\lambda), \tag{3.61}$$

where $\{E(\lambda)\}_{\lambda\in[0,\infty)}$ is the spectral measure of the standard Laplacian on $L^2(\mathbb{R}^3)$.

So far, we have studied the quantized extended free rigid body through the attempt to have simultaneous spectral resolution of the two operators Δ_H and Δ_L on the Hilbert space $L^2(G_{\mathsf{H}})$. The spectra of Δ_H has been determined by the Plancherel formulae for arbitrary extended free rigid bodies. As was seen in the respective cases, the simultaneous spectral resolution of Δ_H and Δ_L with $J_2 = J_3$ are given explicitly in each irreducible representations.

Theorem 2. *For the quantized extended free rigid body, the spectral resolution of Δ_H are given by* Eqs.(3.25), (3.32), (3.45), (3.50), (3.54), *and* (3.60), *according to the signature of the symmetric matrix* H. *If the extended free rigid body is an extended free symmetric top, then the operator Δ_L admits an explicit spectral resolution simultaneous with Δ_H. The spectra of Δ_L are given by* Eqs.(3.28), (3.35), (3.39), (3.43), (3.48), (3.58), *and* (3.61), *in respective representation spaces for Δ_H, where the case of* $\mathrm{sgn}(\mathsf{H}) = (1,1,1)$ *is excluded, since no extended symmetric top is allowed.*

Remark 4. The equations in Theorem 2 can be listed as follows:

$\mathrm{sgn}(\mathsf{H})$	$(3,0,0)$ $(0,0,3)$	$(2,0,1)$ $(1,0,2)$	$(2,1,0)$ $(0,1,2)$	$(1,1,1)$	$(1,2,0)$ $(0,2,1)$	$(0,3,0)$
Δ_H	(3.25)	(3.32)	(3.45)	(3.50)	(3.54)	(3.60)
Δ_L with $J_2 = J_3$	(3.28)	(3.35) (3.39) (3.43)	(3.48)	$(J_2 \neq J_3)$	(3.58)	(3.61)

4. Concluding Remarks

In defining the classical mechanics for an extended free rigid body, we have exchanged the roles of the two first integrals of the Euler equation. In fact, the integral H, which defines the Lie-Poisson structure $\{\cdot,\cdot\}_{\mathsf{H}}$, is of geometric nature, while the other integral L, which stands for the Hamiltonian, is of mechanical nature, contrary to the case of the ordinary free rigid body. This exchange of the roles between the first integrals can be interpreted by the concept of bi-Hamiltonian structures. The classical mechanics for free rigid bodies is studied through bi-Hamiltonian structures, e.g., in [2, 3, 8]. See also Remark 2 in Sec.2.

As for the quantum mechanics, our method of quantization originates in [10, 12, 13]. Though the quantization for an ordinary free rigid body is performed in [10], our study of the quantum mechanics for an extended free rigid body requires representation theory for Lie groups other than $\mathfrak{so}(3)$. There can be, of course, other styles of quantization for free rigid bodies. Nambu treats the quantization of the generalized Hamiltonian dynamics within the Heisenberg picture in [9]. Skrypnyk considers the quantum integrability of the Euler top from the viewpoint of the algebraic relations of the operators in [14]. There is another paper by Komarov and Kuznetsov [5] which treats the quantization of the Euler-Manakov top on the three-dimensional sphere. In comparison with these studies, our study is performed along the line of [10, 12, 13] and results in the explicit simultaneous spectral resolution of the operators Δ_H and Δ_L for an extended free symmetric top, as is shown in Theorem 2. This main result generalizes the explicit spectra for an ordinary free symmetric top, which is described in [10]. To sum up, explicit solutions are found both in classical and quantum mechanics for an extended free symmetric top due to the additional $SO(2)$-symmetry.

Acknowledgements. The second author would like to express his deep gratitude to Professor Tudor Ratiu for significant comments. The authors would also like to thank the referee, who called their attention to some of the references.

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A. Plancherel Formula for the Poincaré Group in two dimensions

The Poincaré group $\mathcal{L}_{\mathrm{e}}(2) := SO_{\mathrm{e}}(1,1) \ltimes \mathbb{R}^2$ can be realized as the matrix group

$$\left\{ (t,x,y) = \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} ; t,x,y \in \mathbb{R} \right\} \tag{A.1}$$

with the multiplication law $(t',x',y') \cdot (t,x,y) = (t'+t, x'+e^{t'}x, y'+e^{-t'}y)$ for $(t,x,y),(t',x',y') \in \mathcal{L}_{\mathrm{e}}(2)$. The Lie group $\mathcal{L}_{\mathrm{e}}(2)$ has the bi-invariant Haar measure $d\mu := dt \wedge dx \wedge dy$, so that it is unimodular.

All irreducible unitary representations of $\mathcal{L}_{\mathrm{e}}(2)$ can be given by the induced representation method. Indeed, the representations

$$U_\lambda = \mathrm{Ind}_{\mathbb{R}^2,\alpha}^{\mathcal{L}_{\mathrm{e}}(2)}, \ \ \alpha = (p,q) \in \mathbb{R}^2, \ \ pq = \lambda, \tag{A.2}$$

with $\lambda \in \mathbb{R}$ exhaust all the inequivalence classes of irreducible unitary representations. Let $\phi_\alpha : \mathbb{R}^2 \ni (x,y) \mapsto e^{\sqrt{-1}(px+qy)} \in U(1)$ be an irreducible representation of $\mathbb{R}^2$ associated with the vector $\alpha \in \mathbb{R}^2$. The representation U_λ can be realized on the Hilbert space $\mathcal{H}_{\phi_\alpha}$ consisting of measurable functions f on $\mathcal{L}_{\mathrm{e}}(2)$ such that

$$f((0,x',y') \cdot (t,x,y)) = \phi_\alpha(x',y')^{-1} f(t,x,y) \tag{A.3}$$

and

$$\int_M |f([t,x,y])|^2 d\mu'([t,x,y]) < \infty, \tag{A.4}$$

where $M = \mathcal{L}_{\mathrm{e}}(2)/\mathbb{R}^2$, where $[t,x,y]$ denotes the image of (t,x,y) under the projection $\mathcal{L}_{\mathrm{e}}(2) \to M$, and where $d\mu'$ is the $\mathcal{L}_{\mathrm{e}}(2)$-invariant measure on M. Note that $|f([t,x,y])|$ is well-defined on M. The representation U_λ is given by

$$U_\lambda((t',x',y'))f(t,x,y) = f((t,x,y) \cdot (t',x',y')^{-1}), \tag{A.5}$$

where $(t,x,y),(t',x',y') \in \mathcal{L}_{\mathrm{e}}(2)$. Note that this group-representation gives rise to the derived representation of the Lie algebra $\mathfrak{so}(1,1) \ltimes \mathbb{R}^2$ as

$$\begin{aligned} U_\lambda\left(\frac{\partial}{\partial t}\right) f &= \frac{\partial f}{\partial t}, \\ U_\lambda\left(\frac{\partial}{\partial x}\right) f &= \sqrt{-1} p e^t f, \\ U_\lambda\left(\frac{\partial}{\partial y}\right) f &= \sqrt{-1} q e^{-t} f, \end{aligned} \tag{A.6}$$

where $f \in \mathcal{H}_{\phi_\alpha}$. The isomorphism between the Lie algebra $\mathfrak{so}(1,1) \ltimes \mathbb{R}^2$ and $\mathbb{R}^3_{\mathsf{H}}$ associated with $\mathsf{H} = \mathrm{diag}(0, J_2, J_3)$, where $J_2 > 0 > J_3$, is given as

$$X_1 = \sqrt{-J_2 J_3}\frac{\partial}{\partial t},\ X_2 = \frac{\sqrt{J_3}}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right),\ X_3 = \frac{\sqrt{-J_2}}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right). \tag{A.7}$$

Since $\mathcal{L}_{\mathrm{e}}(2)$ is diffeomorphic to $\mathbb{R}^3$ and since the Haar measure $d\mu$ is nothing but the Lebesgue measure on $\mathbb{R}^3$, any function $g \in L^2(\mathcal{L}_{\mathrm{e}}(2))$ can be put in the form of the Fourier integral

$$g(t,x,y) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} \hat{g}(s,p,q)e^{-\sqrt{-1}(st+px+qy)}dsdpdq, \tag{A.8}$$

where

$$\hat{g}(s,p,q) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} g(t',x',y')e^{\sqrt{-1}(st'+px'+qy')}dt'dx'dy'. \tag{A.9}$$

Moreover, if $f \in \mathcal{H}_{\phi_\alpha}$ is regarded as a function on $\mathcal{L}_{\mathrm{e}}(2)$, on account of $f(t,x,y) = \phi_\alpha(x,y)^{-1}f(t,0,0)$ from Eq.(A.3), it can be expressed as

$$f(t,x,y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{f}(s)e^{-\sqrt{-1}st}ds\phi_\alpha(x,y)^{-1}, \tag{A.10}$$

where

$$\tilde{f}(s) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t',0,0)e^{\sqrt{-1}st'}dt'. \tag{A.11}$$

We can calculate the Hilbert-Schmidt norm of the multiplication operator of a function $k \in \mathcal{C}_0^\infty(\mathcal{L}_{\mathrm{e}}(2))$ as follows: Since

$$\begin{aligned}
&U_\lambda(k)f|_{(t,x,y)} \\
=&\int_{\mathcal{L}_{\mathrm{e}}(2)} k(t',x',y')U_\lambda(t',x',y')f(t,x,y)dt'dx'dy' \\
=&\int_{\mathcal{L}_{\mathrm{e}}(2)} k(t',x',y')f((t,x,y)\cdot(t',x',y')^{-1})dt'dx'dy' \\
=&\int_{\mathcal{L}_{\mathrm{e}}(2)} k(t',x',y')\phi_\alpha(x-e^{t-t'}x', y-e^{t'-t}y')^{-1}f(t-t',0,0)dt'dx'dy' \\
=&\frac{1}{\sqrt{2\pi}}\int_{\mathcal{L}_{\mathrm{e}}(2)}\int_{\mathbb{R}} k(t',x',y')\phi_\alpha(x-e^{t-t'}x', y-e^{t'-t}y')^{-1}\tilde{f}(s)e^{-\sqrt{-1}s(t-t')}dsdt'dx'dy' \\
=&\frac{1}{\sqrt{2\pi}}\int_{\mathcal{L}_{\mathrm{e}}(2)}\int_{\mathbb{R}} k(t',x',y')e^{-\sqrt{-1}\left\{s(t-t')+p(x-e^{t-t'}x')+q(y-e^{t'-t}y')\right\}}\tilde{f}(s)dsdt'dx'dy'
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{\sqrt{2\pi}}\int_{\mathcal{L}_{\rm e}(2)}\frac{1}{(2\pi)^{\frac{2}{3}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\hat{k}(s',p',q')e^{-\sqrt{-1}\left\{(s'-s)t'+(p'-pe^{t-t'})x'+(q'-e^{t'-t}q)y'\right\}}\\
&\qquad\qquad\qquad\qquad\qquad\times\tilde{f}(s)e^{-\sqrt{-1}(st+px+qy)}ds'dp'dq'dsdt'dx'dy'\\
=&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3}\tilde{f}(s)\hat{k}(s',pe^{t-t'},qe^{t'-t})e^{-\sqrt{-1}(st+px+qy)}e^{-\sqrt{-1}(s'-s)t'}dt'dsds'\\
=&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\tilde{f}(s)\hat{k}(s',pe^{t-t'},qe^{t'-t})e^{-\sqrt{-1}(s'-s)t'}dt'ds'e^{-\sqrt{-1}(st+px+qy)}ds, \qquad \text{(A.12)}
\end{aligned}
$$

its Hilbert-Schmidt norm is

$$
\begin{aligned}
||U_\lambda(k)||_{HS}^2 &= \int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}\hat{k}(s',pe^{t-t'},qe^{t'-t})e^{-\sqrt{-1}(s'-s)t'}dt'\right|^2ds'ds\\
&= \int_{\mathbb{R}^2}|\hat{k}(S,pe^T,qe^{-T})|^2dTdS. \qquad \text{(A.13)}
\end{aligned}
$$

This is because the integral

$$
\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{k}(s',pe^{t-t'},qe^{t'-t})e^{-\sqrt{-1}st'}dt' \qquad \text{(A.14)}
$$

is the Fourier transform of the function $\hat{k}(s',pe^{t-t'},qe^{t'-t})$ of the variable t'. Thus, by the ordinary Plancherel formula on $L^2(\mathbb{R}^2)$, we have

$$
\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\hat{k}(s',pe^{t-t'},qe^{t'-t})e^{\sqrt{-1}st'}dt'\right|^2ds' = \int_{\mathbb{R}}|\hat{k}(s',pe^{t-t'},qe^{t'-t})|^2dt'. \qquad \text{(A.15)}
$$

By the ordinary Parseval-Plancherel formula in commutative Fourier analysis, we get

$$
2\int_{-\infty}^{\infty}||U_\lambda(k)||_{HS}^2d\lambda = \int_{\mathbb{R}^3}|\hat{k}(S,P,Q)|^2dSdPdQ = ||k||_{L^2(\mathcal{L}_{\rm e}(2))}^2. \qquad \text{(A.16)}
$$